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Some results on Ehrhart polynomials of star-shaped triangulations of balls

Takayuki Hibi
日比孝之（北海道大学理学部）

Department of Mathematics
Faculty of Science
Hokkaido University
Kita-ku, Sapporo 060, Japan

In this paper we extend the work [Sta5] and [H4] on Ehrhart polynomials of convex polytopes.

First, we recall what the Ehrhart polynomial of a convex polytope is. Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope, i.e., a convex polytope any of whose vertices has integer coordinates, of dimension d and $\partial\mathcal{P}$ the boundary of $\mathcal{P}$. Given an integer $n > 0$ we write $i(\mathcal{P},n)$ for the number of rational points $(\alpha_1,\alpha_2,\ldots,\alpha_N)$ in $\mathcal{P}$ such that each $n\alpha_i$ is an integer. In other words, $i(\mathcal{P},n) = \#(n\mathcal{P}\cap\mathbb{Z}^N)$. Here $n\mathcal{P} := \{n\alpha; \alpha\in\mathcal{P}\}$ and $\#(X)$ is the cardinality of a finite set X. The systematic study of $i(\mathcal{P},n)$ originated in the work of Ehrhart (cf. [Ehr]), who established that the function $i(\mathcal{P},n)$ possesses the following fundamental properties :

(0.1) $i(\mathcal{P},n)$ is a polynomial in n of degree d.
(Thus $i(\mathcal{P},n)$ can be defined for *every* integer n.)

(0.2) $i(\mathcal{P},0) = 1$.

(0.3) ("loi de réciprocité") $(-1)^d\, i(\mathcal{P},-n) = \#(n(\mathcal{P}-\partial\mathcal{P})\cap\mathbb{Z}^N)$ for every integer $n > 0$.

We say that $i(\mathcal{P},n)$ is the *Ehrhart polynomial* of $\mathcal{P}$.

We define the sequence $\delta_0, \delta_1, \delta_2, \ldots$ of integers by the formula

$$(1-\lambda)^{d+1}\left[1+\sum_{n=1}^{\infty} i(\mathcal{P},n)\lambda^n\right] = \sum_{i=0}^{\infty} \delta_i \lambda^i. \qquad (*)$$

Thus, in particular, $\delta_0 = 1$ and $\delta_1 = \#(\mathcal{P} \cap \mathbb{Z}^N) - (d+1)$. Also, $\delta_i = 0$ for every $i > d$ by, e.g., [Sta3, Corollary 4.3.1], and $\delta_d = \#((\mathcal{P}-\partial\mathcal{P}) \cap \mathbb{Z}^N)$ by (0.3). Moreover, each δ_i is non-negative [Sta1]. When $d = N$, $(\sum_{0 \le i \le d} \delta_i)/d!$ is equal to the volume of $\mathcal{P}$ ([Sta3, Proposition 4.6.30]). We say that the sequence $\delta(\mathcal{P}) := (\delta_0, \delta_1, \ldots, \delta_d)$ which appears in Eq. (*) is the *δ-vector* of $\mathcal{P}$. Some linear inequalities on the δ-vector of $\mathcal{P}$ are known, e.g.,

(1.1) ([Sta5]) If $\delta_j \neq 0$ and $\delta_{j+1} = \delta_{j+2} = \ldots = \delta_d = 0$, then $\sum_{0 \le \ell \le i} \delta_\ell \le \sum_{0 \le \ell \le i} \delta_{j-\ell}$ for every $0 \le i \le [j/2]$.

(1.2) ([H4]) When $(\mathcal{P}-\partial\mathcal{P}) \cap \mathbb{Z}^N$ is non-empty, then $\delta_1 \le \delta_i$ for every $1 \le i < d$.

See also [Sta6], [B-M], [H1], [H2] and [H3] for further results on Ehrhart polynomials of convex polytopes.

On the other hand, let Γ be an integral polyhedral complex in $\mathbb{R}^N$, i.e., a finite set of integral convex polytopes in $\mathbb{R}^N$ such that (a) if $\mathcal{P} \in \Gamma$ and $\mathcal{Q}$ is a face of $\mathcal{P}$, then $\mathcal{Q} \in \Gamma$, and (b) if $\mathcal{P}, \mathcal{P}' \in \Gamma$, then $\mathcal{P} \cap \mathcal{P}'$ is a face of $\mathcal{P}$ and of $\mathcal{P}'$. We write $|\Gamma|$ for the underlying space $\cup_{\mathcal{P} \in \Gamma} \mathcal{P}$ $(\subset \mathbb{R}^N)$ of Γ, and let $\partial|\Gamma|$ be the boundary of $|\Gamma|$ (in the usual topological sense with respect to the relative topology on $|\Gamma|$ inherited from the standard topology on $\mathbb{R}^N$). We call $d := \max\{\dim \mathcal{P}; \mathcal{P} \in \Gamma\}$ the dimension of Γ. In analogy with $i(\mathcal{P},n)$, we define for $n > 0$ $i(\Gamma,n)$ to be the number of rational points $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ in $|\Gamma|$ for which each $n\alpha_i \in \mathbb{Z}$. Thus, thanks to (0.3), we easily see

$$i(\Gamma,n) = \sum_{\mathcal{P} \in \Gamma} (-1)^{\dim \mathcal{P}}\, i(\mathcal{P},-n).$$

Hence $i(\Gamma,n)$ is a polynomial in n of degree d, however, $i(\Gamma,0) = \chi(\Gamma)$, the Euler characteristic of Γ.

Now, suppose that Γ **is an integral polyhedral complex in** $\mathbb{R}^N$ **of dimension** d **such that the underlying space** $|\Gamma|$ **is homeomorphic to the** d**-ball. Then** $\chi(\Gamma) = 1$ **. Hence the** δ**-vector** $\delta(\Gamma) = (\delta_0, \delta_1, \ldots, \delta_d)$ **of** Γ **can be defined by replacing** $i(\mathcal{P}, n)$ **with** $i(\Gamma, n)$ **in Eq. (*). Thus** $\delta_0 = 1$ **,** $\delta_1 = \#(|\Gamma| \cap \mathbb{Z}^N) - (d+1)$ **and** $\delta_d = \#((|\Gamma| - \partial|\Gamma|) \cap \mathbb{Z}^N)$ **. Here** $\partial|\Gamma|$ **is the boundary, which is homeomorphic to the** $(d-1)$**-sphere, of** $|\Gamma|$ **. Also, each** δ_i **is non-negative [Sta4].**

We are now in the position to state our main result in this paper.

THEOREM. Let Γ **be an integral polyhedral complex in** $\mathbb{R}^N$ **of dimension** d **such that the underlying space** $|\Gamma|$ **is homeomorphic to the** d**-ball. Suppose that** $(|\Gamma| - \partial|\Gamma|) \cap \mathbb{Z}^N$ **is non-empty, i.e.,** $\delta_d \neq 0$ **, and that** $|\Gamma|$ **is star-shaped relative to some** $\alpha \in (|\Gamma| - \partial|\Gamma|) \cap \mathbb{Z}^N$ **, i.e., for each** β **in** $|\Gamma|$ **, the open line segment** $\{ (1-t)\alpha + t\beta \,;\, 0 < t < 1 \}$ **from** α **to** β **lies in** $|\Gamma| - \partial|\Gamma|$ **. Let** $\delta(\Gamma) = (\delta_0, \delta_1, \ldots, \delta_d)$ **be the** δ**-vector of** Γ **.**

(2.1) We have the lower bound inequality

$$\delta_1 \leq \delta_i$$

for every $1 \leq i < d$ **.**

(2.2) Moreover, the linear inequality

$$\delta_0 + \delta_1 + \ldots + \delta_i \leq \delta_d + \delta_{d-1} + \ldots + \delta_{d-i}$$

holds for every $0 \leq i \leq [d/2]$ **.**

The proof of (2.1) is based on the same combinatorial technique as in [H4]. On the other hand, in [Sta5], the canonical ideal $\Omega(A(\mathcal{P}))$ **of a Cohen-Macaulay integral domain** $A(\mathcal{P})$ **associated with a convex polytope** $\mathcal{P}$ **plays an essential role for the proof of (1.1).**

More generally, in [Sta$_4$, p.202], given an integral polyhedral complex Γ **in** $\mathbb{R}^N$ **, a noetherian graded ring** $A(\Gamma)$ **is defined as follows. Let** $X_1, \ldots, X_N$ **and** T **be indeterminates over a field** k **. A basis of** $A(\Gamma)$ **as a vector space over** k **consists of** 1 **together with all monomials** $X^{\alpha}T^n = X_1{}^{\alpha_1} \ldots X_N{}^{\alpha_N}T^n$ **, where** $n > 0$ **is an integer and** $\alpha = (\alpha_1,\ldots,\alpha_N) \in (n|\Gamma| \cap \mathbb{Z}^N)$ **. In** $A(\Gamma)$ **, multiplication of two monomials** $X^{\alpha}T^n$ **and** $X^{\beta}T^m$ **is defined by (i)** $(X^{\alpha}T^n)(X^{\beta}T^m) = X^{\alpha+\beta}T^{n+m}$ **if** $\alpha \in n\mathcal{P}$ **and** $\beta \in m\mathcal{P}$ **for some** $\mathcal{P} \in \Gamma$ **, and (ii)** $(X^{\alpha}T^n)(X^{\beta}T^m) = 0$ **otherwise. (Note that** $A(\Gamma)$ **is never an integral domain unless** $|\Gamma| = \mathcal{P}$ **for some** $\mathcal{P} \in \Gamma$ **.) We define a grading on** $A(\Gamma)$ **by setting** $\deg(X^{\alpha}T^n) = n$ **. Then the Hilbert function** $H(A(\Gamma),n)$ **of** $A(\Gamma)$ **is equal to** $i(\Gamma,n)$ **for every** $n > 0$ **.**

Thanks to [Sta$_4$, Lemma 4.6], if $|\Gamma|$ **is homeomorphic to the** d**-ball, then the algebra** $A(\Gamma)$ **is Cohen-Macaulay. Moreover, the canonical ideal** $\Omega(A(\Gamma))$ **of** $A(\Gamma)$ **can be expressed easily by virtue of Hochster's theorem ([Sta$_2$, p.81]). The key lemma for our proof of (2.2) is the existence of an integral polyhedral complex** Γ' **in** $\mathbb{R}^N$ **with** $|\Gamma| = |\Gamma'|$ **such that the canonical ideal** $\Omega(A(\Gamma'))$ **possesses a certain non-zero divisor on** $A(\Gamma')$ **which is required in, e.g., [H$_5$, Proposition (1.3)].**

EXAMPLE. Let $\mathcal{P} \subset \mathbb{R}^3$ **be the simplex with vertices** $A = (1,0,0)$ **,** $B = (0,1,0)$ **,** $C = (0,0,1)$ **and** $D = (-1,-1,-1)$ **. Also, let** $\mathcal{Q} \subset \mathbb{R}^3$ **be the simplex whose vertices are** A **,** B **,** C **and** $E = (1,1,0)$ **. We define** Γ **to be the integral polyhedral complex in** $\mathbb{R}^3$ **of dimension** 3 **which consists of** $\mathcal{P}$ **,** $\mathcal{Q}$ **, all faces of** $\mathcal{P}$ **and all faces of** $\mathcal{Q}$ **. Then the underlying space** $|\Gamma|$ **is homoemorphic to the** 3**-ball, and** $(|\Gamma| - \partial|\Gamma|) \cap \mathbb{Z}^3 = \{ (0,0,0) \}$ **. However,** $|\Gamma|$ **is not star-shaped relative to the origin of** $\mathbb{R}^3$ **. We have** $\delta(\Gamma) = (1,2,1,1)$ **, which fails to satisfy the inequality in (2.1) for** $i = 2$ **and that in (2.2) for** $i = 1$ **.**

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